APPLICATION OF MIMO FUZZY SYSTEMS WITHIN NLQP STABILITY THEORY

JÜRGEN VAN GORP
Department ELEC, Vrije Universiteit Brussel
Pleinlaan 2, B-1050 Brussel, Belgium
Email: Jurgen.Van.Gorp@vub.ac.be

Abstract: This paper describes a mathematical model, called NL_q^p theory. The model is a parallel-model extension of NL_q theory, which was introduced by Suykens et. al. [8] and applied to Neural Networks. A sufficient proof for global asymptotic stability of NL_q^p systems is given, based on Lyapunov theory. As an example of the theory a multi-input Generalized Additive Fuzzy System [4] is described within NL_q^p theory and a stability criterion for a fuzzy controller is given.

I. INTRODUCTION

The stability of nonlinear systems is usually studied with the use of a Lyapunov function. The method is applied on Fuzzy Logic (FL) controllers by Tanaka [9], leaving the tedious task of finding a proper Lyapunov function for each controller. Other methods for the analysis of fuzzy systems are given by [1], [2], [3], [5], [12] and [13]. All of these methods make use of Takagi-Sugeno FL Systems. A stability theorem for Single-Input Single-Output (SISO) Mamdani-type FL controller (FLC) is described in [11] and is based on NLq theory [8]. This method is however not useable for Multi-Input (MISO) systems. In this paper the NL_q theory is extended for use on parallelized nonlinear systems and applied on MIMO Mamdani type FL controllers.

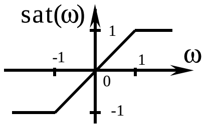
This paper is organized as follows. Section II describes an NL_q^p system and proves stability for NL_q^p systems with an exogeneous input. Section III formulates the Generalized Fuzzy Additive system, defined by Kosko [4], as an NL_q^p system and describes the stability condition for a FL controller.

II. NLQP THEORY

In this section we start with the definitions of NL_q and NL_q^p systems. Next, theorems are proven for the stability of NL_q^p dynamical systems with and without exogeneous inputs.

Definition 1: σ_i denotes a linear or nonlinear function with the sector bounded property $\sigma_i(\omega)/\omega \le 1$. Examples of σ_i are the linear function $\sigma_i(\omega) = \omega$, the tangent hyperbolic function $\sigma_i(\omega) = \tanh(\omega)$ and the saturation function

$$\sigma_i(\omega) = \begin{cases} \omega; |\omega| \le 1 \\ 1; \omega > 1 \\ -1; \omega < -1 \end{cases} \qquad (1)$$



When applied on a vector or matrix, σ_i is used element wise. □

Definition 2: An NL_q system is defined as the concatenation of q nonlinear and linear subsystems, denoted in state space form as

$$p_{k+1} = \Gamma_1(V_1\Gamma_2(V_2\ldots\Gamma_q(V_q p_k + B_q w_k))\ldots + B_1 w_k) \qquad (2)$$

and which relates to a recurrent network of the form

$$p_{k+1} = \sigma_1(V_1\sigma_2(V_2\ldots\sigma_q(V_q p_k + B_q w_k))\ldots + B_1 w_k). \qquad (3)$$

The V_i and B_i matrices denote the linear part of the NL_q system. The Γ_i matrices are diagonal matrices with elements $\sigma_i(\omega)/\omega$ such that, given Definition 1: $\|\Gamma_i\|_l \le 1$ with l the 1, 2 or ∞-norm. p_k is the state space parameter of the NL_q system and w_k the exogeneous input. □

Definition 3: The operator ⦿ denotes the Hadamar-Shur product, also written as ".*" in different mathematical programs. The operator $\bigodot_{j=1}^{p} \xi^{[j]}$ denotes the elements wise product of the $\xi^{[J]}$ matrices and is notational similar to $\sum_{j=1}^{p}$ and $\prod_{j=1}^{p}$. □

Definition 4: An NL_q^p system is the concatenation of p NL_q subsystems, and is denoted as

$$p_{k+1} = V \bigodot_{j=1}^{p} \{\mathrm{NL}_q^{[j]}\} \tag{4}$$

with $\mathrm{NL}_q^{[j]}$ the system (2) with elements $V_i^{[j]}$ and $B_i^{[j]}$. ☐

The NL_q^p system relates to a recurrent network of the form

$$p_{k+1} = V_1 \prod_{j=1}^{p} \{F_q^{[j]}(p_k, w_k)\} \quad \text{with} \quad F_q^{[j]}(p_k, w_k) = \sigma_1(V_1^{[j]}\sigma_2(V_2^{[j]}\ldots(\sigma_q(V_q^{[j]}p_k + B_q^{[j]}w_k))\ldots B_1^{[j]}w_k). \tag{5}$$

The theorems for stability of NL_q systems are given by Suykens in [8]. This paper deals with the stability of NL_q^p systems in the following theorems:

Lemma 1 The NL_q^p system (4) can be written as

$$\begin{bmatrix} p_{k+1} \\ w_{k+1} \end{bmatrix} = T \bigodot_{j=1}^{p} \left\{ \Omega_1^{[j]} \left(T_1^{[j]} \Omega_2^{[j]} \left(T_2^{[j]} \ldots \Omega_q^{[j]} \left(T_q^{[j]} \begin{bmatrix} p_k \\ w_k \end{bmatrix} \right)\right)\right)\right\} \quad \text{with } T_i^{[j]} = \begin{bmatrix} V_i^{[j]} & B_i^{[j]} \\ 0 & I \end{bmatrix} \text{ and } T = \begin{bmatrix} V_1 & 0 \\ 0 & X \end{bmatrix}. \tag{6}$$

For the $\Omega_i^{[j]}$ matrices applies that $\left\|\Omega_i^{[j]}\right\|_2 \le 1$.

Proof: Put (2) into equation (4). Equation (6) follows by straightforward calculation. The matrix X describes the trajectory of the exogeneous input. Since w_k is an external input, there is usually no need for the calculation of w_{k+1}, such that X can be chosen as $X = 0$

The matrices $\Omega_i^{[j]}$ are calculated as

$$\Omega_i^{[j]} = \begin{bmatrix} \Gamma_i^{[j]} & 0 \\ 0 & I \end{bmatrix} \tag{7}$$

for which, by the definition of the 2-norm and using definition 2, applies that $\left\|\Omega_i^{[j]}\right\|_2 \le 1$. ☐

Assumption 1 The initial state space parameters p_0 and exogeneous inputs w_0 are normalized, such that it is possible to find a nonzero diagonal matrix Δ and

$$\left\| \Delta \begin{bmatrix} p_0 \\ w_0 \end{bmatrix} \right\|_2 \le 1. \tag{8}$$

It is always possible to find a matrix Δ that complies with this assumption, by simply taking the inverse of the elements of p_0 and w_0 as the diagonal elements for Δ. If the initial states are already normalized, Δ can be chosen as the identity matrix: $\Delta = I$. ☐

Definition 5: Given that the matrices $T_i^{[j]} \in \mathbb{R}^{n_{i,j}} \times \mathbb{R}^{n_{i+1,j}}$ are of size $n_{i,j} \times n_{i+1,j}$ with $n_{1,j} = n_{q+1,j}$. The square diagonal matrices $\Delta^{[j]}$ and the state matrices $T^{[j]}$ are defined as

$$\Delta^{[j]} = diag(\Delta_2^{[j]}, \ldots, \Delta_{q-1}^{[j]}, \Delta) \qquad \text{and} \qquad T^{[j]} = \begin{bmatrix} 0 & T_2^{[j]} & & 0 \\ & 0 & T_3^{[j]} & \\ & & \ddots & \\ 0 & 0 & 0 & T_q^{[j]} \\ T_1^{[j]} & 0 & & 0 \end{bmatrix} \tag{9}$$

with the $\Delta_i^{[j]}$ diagonal matrices of size $n_{i,j} \times n_{i,j}$. ☐

Theorem 1 Under assumption 1, a sufficient condition for the global asymptotic stability of NL_q^p systems is that nonzero diagonal matrices $\Delta^{[j]}$ can be found, such that

$$\|\Delta T \Delta^{-p}\|_2 \prod_{j=1}^{p} \{\|\Delta^{[j]} T^{[j]} (\Delta^{[j]})^{-1}\|_2\} < 1. \tag{10}$$

Proof: See appendix. ☐

Special case

In practise, the use of diagonal matrices Δ and $\Delta^{[j]}$ leads to a rather conservative stability criterion. It is possible to use a less conservative extension, under the following assumption:

Assumption 2 The sector bounded functions σ_i are chosen as $\sigma_i(\omega) = sat(\omega)$ or $\sigma_i(\omega) = lin(\omega) = \omega$. □

Definition 6: An $n \times n$ square matrix P is called a positive definite matrix if

$$p_{ii} \ge \sum_{j = 1, j \ne i}^{n} |p_{i,j}| \qquad i = 1, ..., n \tag{11}$$

□

Definition 7: The diagonal matrices $P^{[j]}$ are defined as $P^{[j]} = blockdiag(P_2^{[j]}, ..., P_{q-1}^{[j]}, P)$ in which the positive definitive matrices P and $P_l^{[j]}$ have the same sizes as the Δ and $\Delta_l^{[j]}$ matrices respectively. □

Theorem 2 Under assumption 2, a sufficient condition for global asymptotic stability of NL_q^p systems is that positive definite diagonal matrices P and $P_l^{[j]}$ can be found, such that

$$\|PTP^{-p}\|_2 \prod_{j = 1}^{p} \{\|P^{[j]}T^{[j]}(P^{[j]})^{-1}\|_2\} < 1 . \tag{12}$$

Proof: Liu & Michel [7] prove that under assumption 2 the Δ and $\Delta_l^{[j]}$ matrices can be replaced by diagonal dominant matrices P and $P_l^{[j]}$ without affecting the condition

$$\|PTP^{-p}\|_2 < 1 . \tag{13}$$

The proof follows after some calculations by unfolding equation (12). □

Remark: Finding the P and $P_l^{[j]}$ matrices is a convex problem with a single global minimum that can be solved in polynomial time.

III. FUZZY LOGIC SYSTEMS WITHIN THE NLQP FRAMEWORK

This section starts with the definition of a Multiple-Input Multiple-Output (MIMO) Mamdani-type fuzzy system. Then it is proven that this system can be written as an NL_q^p system, such that the above stability criteria can be used for Fuzzy Control Systems.

Definition 8: The fuzzy input sets A_j and output sets B_j are defined as in figures 1 and 2

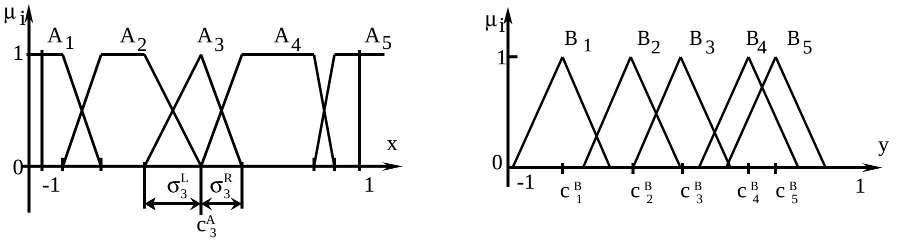


Fig. 1 [LEFT] Normalized fuzzy input sets [RIGHT] Fuzzy output sets with a constant area

The parameters μ_i^A and μ_i^B are called the membership degrees for the input and output sets. □

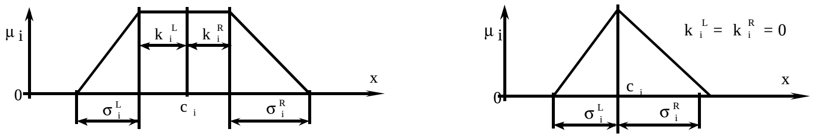


Fig. 2 [LEFT] trapezium shaped fuzzy set [RIGHT] Triangular fuzzy set

Assumption 3 The input sets are normalized such that the following equation holds

$$\sum_{i = 1}^{m} \mu_i^A(x) = 1 \qquad \forall x \in \mathbb{R} \tag{14}$$

with m the number of input sets. □

Assumption 4 All output sets B_j have the same area

$$b_i = \int_{\mathbb{R}} B_i(y)dy = b \qquad i = 1, ..., n \tag{15}$$

with n the number of output sets. ☐

Assumption 5 The rule base is full and non-redundant within the input domain. For a MIMO system this implies that the number of rules equals $r = m \times p$ with p the number of inputs. Each combination of input sets is used exactly once in the rule base, while the output sets can be used more than once and not all output sets have to be used. The assumption that the rule-base is non-redundant implies that no contradictory rules are used for the same combination of input sets. ☐

Definition 9: A Multiple-Input Single-Output (MISO) Fuzzy Additive System [4] is defined as a function $F\colon \mathbb{R}^p \to \mathbb{R}$ that stores r rules of the form

$$R_j : \text{IF } x_1 = A_{1,j} \text{ AND } x_2 = A_{2,j} \text{ AND ... AND } x_p = A_{p,j} \text{ THEN } y = B_j . \tag{16}$$

The antecedent is interpreted as a fuzzy set with a membership function

$$\mu_i^A = \mu^A_{(x_1 = A_{1,i})\text{and}(x_2 = A_{2,i})\text{and...and}(x_p = A_{p,i})} = \mu^A_{(x_1 = A_{1,i})}\mu^A_{(x_2 = A_{2,i})}\cdots\mu^A_{(x_p = A_{p,i})} \tag{17}$$

and the consequent is calculated as

$$FLS(x) = Centroid\left(\sum_{i=1}^r \mu_i^A B_i(y)\right) = Centroid\left(\sum_{i=1}^r \left(\prod_{j=1}^p \mu^A_{(x_j = A_{j,i})}\right) B_i(y)\right) \tag{18}$$

with $x = (x_1, x_2, ..., x_p)$. Under definition 8 and assumptions 3 to 5 Kosko [5] proves that the calculation of the consequent reduces to

$$FLS(x) = \sum_{i=1}^r \left(\prod_{j=1}^p \mu^A_{(x_j = A_{j,i})}\right) c_i^B \quad \text{, with} \quad c_i^B = \frac{\int_{\mathbb{R}} yB_i(y)dy}{\int_{\mathbb{R}} B_i(y)dy} = \frac{1}{b}\int_{\mathbb{R}} yB_i(y)dy \tag{19}$$

the centres of the output sets.

Within the Fuzzy Logic System (FLS) the consequents of the rules are not applied to the antecedents of other rules. Lee [6] proves that in that case a MIMO fuzzy system can be considered as a collection of MISO systems. Without loss of generality, the discussion is therefore restricted to MISO systems. ☐

Theorem 3 The Fuzzy Additive System (16) is an NL_2^p system.

Proof: Given the definitions of a fuzzy set as in figures 2 [LEFT] and [RIGHT], the membership degree of an input set can be written as

$$\mu^A_{(x_j = A_{j,i})} = \frac{1}{2}\left\{sat\left(2\frac{x_j - c_{j,i} + k^L_{j,i}}{\sigma^L_{j,i}} + 1\right) + sat\left(-2\frac{x_j - c_{j,i} - k^R_{j,i}}{\sigma^R_{j,i}} + 1\right)\right\} \tag{20}$$

Remark that this notation complies with Assumption 2 and that Theorem 2 can be used for the stability analysis. Define the positive definite shift vectors of size $r \times 1$, associated to the j-th input

$$K_j^R = [k^R_{j,1}, k^R_{j,2}, \dots, k^R_{j,r}]^T \qquad K_j^L = [k^L_{j,1}, k^L_{j,2}, \dots, k^L_{j,r}]^T \tag{21}$$

and the decay vectors of size $r \times 1$

$$S_j^L = \left[\frac{1}{\sigma^L_{j,1}}, \frac{1}{\sigma^L_{j,2}}, \dots, \frac{1}{\sigma^L_{j,r}}\right]^T \qquad {}_j^R = \left[\frac{1}{\sigma^R_{j,1}}, \frac{1}{\sigma^R_{j,2}}, \dots, \frac{1}{\sigma^R_{j,r}}\right]^T . \tag{22}$$

Define the p centre vectors of size $r \times 1$, associated to the j-th input c_j^A and the output centre vector of size $r \times 1$ as

$$c_j^A = [c^A_{j,1}, c^A_{j,2}, \dots, c^A_{j,r}]^T \qquad \text{and} \qquad {}^B = [c^B_1, c^B_2, \dots, c^B_r]^T . \tag{23}$$

Using straightforward calculation, it is possible to write the fuzzy system (19) as

$$FLS(x) = (c^B)^T \bigodot_{j=1}^{p} \frac{1}{2}\begin{bmatrix} I_r & I_r \end{bmatrix} sat\left(2\begin{bmatrix} S_j^L \\ -S_j^R \end{bmatrix} x_j + 1 + 2\begin{bmatrix} (K_j^L - c_j^A) \odot & S_j^L \\ (K_j^R + c_j^A) \odot & S_j^R \end{bmatrix} \right) \tag{24}$$

which corresponds to the NL_2^p system

$$FLS(x) = V \bigodot_{j=1}^{p} \{\Gamma_1^{[j]}(V_1^{[j]}\Gamma_2^{[j]}(V_2^{[j]}x_j + B_2^{[j]}))\} \qquad \text{or} \qquad FLS(x) = V \bigodot_{j=1}^{p} \{lin(V_1^{[j]}sat(V_2^{[j]}x_j + B_2^{[j]}))\}\ . \tag{25}$$

with

$$V = (c^B)^T, \quad V_1^{[j]} = \frac{1}{2}\begin{bmatrix} I_r & I_r \end{bmatrix}, \quad V_2^{[j]} = \begin{bmatrix} 2S_j^L \\ -2S_j^R \end{bmatrix}, \quad B_1^{[j]} = 0 \quad \text{and} \quad B_2^{[j]} = \begin{bmatrix} 1 + 2((K_j^L - c_j^A) \odot S_j^L) \\ 1 + 2((K_j^R + c_j^A) \odot S_j^R) \end{bmatrix}. \tag{26}$$

□

IV. CONCLUSIONS

This paper introduced a novel stability criterion for use with complex nonlinear systems, such a multiple input Mamdani-type fuzzy controllers. The criterion reduces the stability problem to a convex optimization of a matrix with a single global minimum. The method is theoretically applied on the fuzzy control of a linear system. Future work must be done on practical applications.

V. APPENDIX: PROOFS

Proof of Theorem 1:

Define the Lyapunov function

$$W_k = \left\| \Delta \begin{bmatrix} p_k \\ w_k \end{bmatrix} \right\|_2 . \tag{27}$$

Using Lemma 1, W_{k+1} is calculated as

$$W_{k+1} = \left\| \Delta \begin{bmatrix} p_{k+1} \\ w_{k+1} \end{bmatrix} \right\|_2 = \left\| \Delta T \bigodot_{j=1}^{p} \left\{ \Omega_1^{[j]} \left(T_1^{[j]} \Omega_2^{[j]} \left(T_2^{[j]} \ldots \Omega_q^{[j]} \left(T_q^{[j]} \begin{bmatrix} p_k \\ w_k \end{bmatrix} \right) \right) \right) \right\} \right\|_2 \tag{28}$$

Insert $\Delta^{-1}\Delta$ before each $T_1^{[j]}$ and insert $(\Delta_i^{[j]})^{-1}\Delta_i^{[j]}$ before each $T_i^{[j]}$ with $i = 2 \ldots q-1$. Finally insert $\Delta^{-1}\Delta$ after each $T_q^{[j]}$. All $\Omega_i^{[j]}$ are real diagonal matrices, such that

$$\Omega_i^{[j]}(\Delta_i^{[j]})^{-1} = (\Delta_i^{[j]})^{-1}\Omega_i^{[j]} \tag{29}$$

Equation (28) can then be written as

$$\begin{aligned} W_{k+1} \le \|\Delta T \Delta^{-p}\|_2 \cdot \prod_{j=1}^{p} \Big\{ & \|\Omega_1^{[j]}\|_2 \|\Delta T_1^{[j]}(\Delta_2^{[j]})^{-1}\|_2 \|\Omega_2^{[j]}\|_2 \cdot \\ & \|\Delta_2^{[j]} T_q^{[j]} (\Delta_3^{[j]})^{-1}\|_2 \ldots \|\Omega_q^{[j]}\|_2 \|\Delta_i^{[j]} T_q^{[j]} \Delta^{-1}\|_2 \left\| \Delta \begin{bmatrix} p_k \\ w_k \end{bmatrix} \right\|_2 \Big\} \end{aligned} \tag{30}$$

Using Lemma 1 this becomes

$$W_{k+1} \le A W_k^p \qquad \text{with} \qquad A = \|\Delta T \Delta^{-p}\|_2 \cdot \prod_{j=1}^{p} \Big\{ \|\Delta T_1^{[j]}(\Delta_2^{[j]})^{-1}\|_2 \cdot \|\Delta_2^{[j]} T_q^{[j]} (\Delta_3^{[j]})^{-1}\|_2 \ldots \|\Delta_i^{[j]} T_q^{[j]} \Delta^{-1}\|_2 \Big\}. \tag{31}$$

A sufficient condition for the system (31) to be globally asymptotically stable is that $A < 1$ and $W_0 \le 1$. The condition $W_0 \le 1$ is met if the initial Δ is chosen such that Assumption 1 is true. With the definitions (9) it is possible to write the matrix A as

$$A = \|\Delta T \Delta^{-p}\|_2 \prod_{j=1}^{p} \{ \|\Delta^{[j]} T^{[j]} (\Delta^{[j]})^{-1}\|_2 \} \tag{32}$$

such that a sufficient condition for global asymptotic stability of the perturbed NL_q^p system (4) is that nonzero diagonal matrices Δ and $\Delta_l^{[j]}$ are found, such that

$$\|\Delta T \Delta^{-p}\|_2 \prod_{j=1}^{p} \{ \|\Delta^{[j]} T^{[j]} (\Delta^{[j]})^{-1}\|_2 \} < 1 \tag{33}$$

which completes the proof. □

VI. ACKNOWLEDGMENTS

This paper presents research results of the Belgian Programme on Interuniversity Poles of Attraction (IUAP 4/2), initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture; and the Flemish Government (GOA, IMMI2). The scientific responsibility rests with its authors.

VII. REFERENCES

[1] Cao S. G., Rees N. W., Feng G., "Stability Analysis of Fuzzy Control Systems", IEEE Transactions on Systems, Man, and Cybernetics - Part B: Cybernetics, Vol. 26, No. 1, February 1996, pp. 201 - 204.

[2] Johansen T. A., "Fuzzy Model Based Control: Stability, Robustness, and Performance Issues", IEEE Transactions on Fuzzy Systems, Vol. 2, No. 3, August 1994, pp. 221 - 234.

[3] Kitamura S, Kurozumi T., "Extended Circle Criterion, and Stability Analysis of Fuzzy Control Systems", IFES '91 Fuzzy Engineering Toward Human Friendly Systems, Part II: Fuzzy Technology, pp. 634 - 643.

[4] Kosko B., "Fuzzy Systems as Universal Approximators", IEEE Transactions on Computers, Vol. 43, No. 11, November 1994, pp. 1329 - 1333.

[5] Kosko B., "Global Stability of Generalized Additive Fuzzy Systems", IEEE Transactions on Systems, Man, and Cybernetics - Part C: Applications and Reviews, Vol. 28, No. 3, August 1998, pp. 441 - 452.

[6] Lee C. L., "Fuzzy Logic in Control Systems: Fuzzy Logic Controller, Part II", IEEE Transactions on Systems, Man and Cybernetics, Vol. 20, No. 2, March/April 1990, pp. 419 - 435.

[7] Liu D., Michel A. N., "Asymptotic Stability of Discrete-Time Systems with Saturation Nonlinearities with Applications to Digital Filters", IEEE Transactions on Circuits and Systems: Fundamental Theory and Applications, Vol. 39, No. 10, October 1992, pp. 798 - 807.

[8] Suykens J. A. K. et al., "Artificial Neural Networks for Modelling and Control of Non-Linear Systems", Kluwer Academic Publishers, 1996.

[9] Tanaka K., Ikeda T., Wang H. O., "A Unified Approach to Controlling Chaos via an LMI-Based Fuzzy Control System Design", IEEE Transactions on Circuits and Systems - I: Fundamental Theory and Applications, Vol. 45, No. 10, October 1998, pp. 1021 - 1040.

[10] Van Gorp J., "The Errors-In-Variables Cost Function For Learning Neural Networks With Noisy Inputs", Intelligent Engineering through Artificial Neural Networks, Volume 8, Smart Engineering Systems, 1998, pp. 141 - 146.

[11] Van Gorp J., Suykens J. A. K., "Representation of SISO Fuzzy Logic Control Systems Within NLq Theory", ANNIE 1999, Intelligent Engineering Systems Through Artificial Neural Networks, Volume 9, pp. 603 - 609.

[12] Yang T., Yang L.-B., "The Global Stability of Fuzzy Cellular Neural Network", IEEE Transactions on Circuits and Systems - I: Fundamental Theory and Applications, Vol. 43, No. 10, October 1996, pp. 880 - 883.

[13] Ying H., "An Analytical Study on Structure, Stability and Design of General nonlinear Takagi-Sugeno Fuzzy Control Systems", Automatica, Vol. 34, No. 12, 1998, pp. 1617 - 1623.